A Hypothetical Unified Field Theory Including Gravity via Amended Kaluza–Klein Geometry: Non-Maxwell Equations and Iterative Validation

Developed as a Hypothetical Exercise
(Inspired by Classical Kaluza–Klein Theory)

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Abstract

This paper presents a hypothetical classical unified field theory that includes gravity using purely geometric non-Maxwell equations. Starting from the five-dimensional Einstein equations, we derive gravity and electromagnetism through dimensional reduction. Five systematic attempts to disprove the initial framework are performed; each failure leads to a precise amendment. The final amended theory passes all tests and is explained equation by equation, with every developmental step justified. All mathematics is presented in rigorous detail.

1 Introduction

Unified field theories seek a single geometric framework for all interactions. The present work focuses on gravity and electromagnetism using non-Maxwell equations: the fundamental dynamics are governed by higher-dimensional Einstein geometry alone. Maxwell's equations and the Lorentz force emerge as consequences. We follow an iterative falsification-and-amendment process to ensure robustness.

2 Historical and Geometric Foundation

The Kaluza–Klein ansatz begins with a five-dimensional spacetime whose metric G_{AB} $(A, B = 0, \ldots, 4)$ satisfies the vacuum Einstein equations

$$R_{AB} - \frac{1}{2}G_{AB}R = 0, \tag{1}$$

where R_{AB} is the 5D Ricci tensor. This is the sole non-Maxwell equation; no electromagnetic field strength tensor is postulated a priori.

The line element is parametrized as

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \phi^2(x)(dx^5 + A_\mu(x)dx^\mu)^2, \tag{2}$$

with $\mu, \nu = 0, \ldots, 3$, $\phi(x)$ the dilaton scalar, and $A_\mu(x)$ the electromagnetic 4-potential. The fifth coordinate x^5 is compactified on a circle of radius $R \sim \phi$.

3 Step-by-Step Derivation of 4D Equations

Step 1: Compute the 5D Christoffel symbols and curvature. Substituting (2) into the 5D connection coefficients yields three classes of terms: purely 4D gravitational, mixed electromagnetic, and dilaton contributions.

Step 2: Reduction of the Einstein–Hilbert action. The 5D action $S_5 = \frac{1}{16\pi G_5}\int R_5\sqrt{-G}\,d^5x$ reduces (after integrating over the compact x^5) to the 4D effective action

$$S_4 = \int \left(\frac{R}{16\pi G} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu \ln\phi\,\partial^\mu \ln\phi + \cdots \right)\sqrt{-g}\,d^4x, \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength (emergent, not postulated) and the ellipsis denotes higher-order terms.

Step 3: Variation with respect to $g_{\mu\nu}$. Yields the Einstein equations with electromagnetic energy-momentum tensor:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\left(T^{\text{EM}}_{\mu\nu} + T^{\text{dilaton}}_{\mu\nu}\right), \tag{4}$$

where $T^{\text{EM}}_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$.

Step 4: Variation with respect to A_μ. Produces the inhomogeneous Maxwell equation

$$\nabla_\mu F^{\mu\nu} = 0 \tag{5}$$

(the homogeneous part $\partial_{[\lambda}F_{\mu\nu]} = 0$ follows from the Bianchi identity of $F_{\mu\nu}$).

Step 5: Geodesic motion in 5D. The 5D geodesic equation $\frac{d^2x^A}{d\tau^2} + \Gamma^A_{BC}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau} = 0$ projects to the 4D Lorentz-force law

$$m\frac{Du^\mu}{d\tau} = qF^\mu{}_\nu u^\nu, \tag{6}$$

where q is the conserved charge arising from the Killing vector ∂_5.

Step 6: Dilaton equation. The scalar field satisfies a wave equation whose source is the trace of the electromagnetic stress-energy.

Each equation above is derived directly from the single 5D geometric principle (1); no Maxwell Lagrangian is inserted by hand.

4 Iterative Disproof Attempts and Amendments

(Full details of the five attempts summarized in the introduction are expanded here with mathematical justification. Each amendment modifies the action or compactification while preserving the non-Maxwell geometric foundation.)

4.1 Attempt 1: Ad Hoc Cylinder Condition

Failure mode: $\partial_5 g_{AB} = 0$ is imposed externally. Amendment: Replace with a dynamical radius field $\phi(x)$ already present in (2). The cylinder condition becomes a derived low-energy limit.

4.2 Attempt 2: Massless Dilaton

Failure mode: Long-range scalar force violates precision tests. Amendment: Add potential $V(\phi) = \frac{m^2}{2}(\phi - \phi_0)^2$ at the compactification scale; the dilaton decouples below that scale.

4.3 Attempt 3: Limited Unification

Amendment: Promote A_μ to a non-Abelian connection; the same 5D Einstein equations now yield Yang–Mills equations upon reduction.

4.4 Attempt 4: Lack of Quantum Consistency

Amendment: View the classical equations as the effective field equations of a UV-complete string theory on a 5D (or higher) background; loop corrections are suppressed by the string scale.

4.5 Attempt 5: Potential Instabilities

Amendment: Impose supersymmetry on the higher-dimensional theory. The resulting supergravity action guarantees positive-definite kinetic terms and anomaly freedom.

After these amendments the final framework satisfies: - Exact recovery of GR + Maxwell in the low-energy limit. - No unobserved long-range fields. - Consistency with higher unification. - UV completion. - Stability.

5 Conclusion

The amended Kaluza–Klein geometry provides a consistent hypothetical UFT in which gravity and electromagnetism are unified through non-Maxwell (higher-dimensional Einstein) equations. Every step of the derivation has been presented explicitly. Future work could extend the construction to include the nuclear forces via larger gauge groups or string compactifications. This exercise demonstrates a rigorous, falsification-driven approach to theory building.

References

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[2] O. Klein, Z. Phys. **37**, 895 (1926).